MATH 112A Review: Line Integrals, Surface Integrals, Parametrization of Curves

1. Let $F(x,y) = (x^2+y^2, 0)$ and consider the curve C that lies on the unit circle and $x, y \geq 0$ with counterclockwise orientation. Compute

$$\int_C F(\vec{r}) \cdot d\vec{r}(t).$$

> **Solution:** We set $\vec{r}(t) = (\cos t, \sin t)$ with $r \in [0, \pi/2]$. Then, $\vec{r}\,'(t) = (-\sin t, \cos t)$ and $F(\vec{r}) = (1, 0)$. Then,
>
> $$\int_C F(\vec{r}) \cdot d\vec{r}(t) = \int_0^{\pi/2} -\sin t dt = 0 - 1.$$

2. Let S be the collections of points in the xy-plane that satisfy $x^2 + y^2 \leq 1$. Evaluate

$$\iint_S F(x,y,z) \cdot d\vec{S},$$

where $F(x,y,z) = (xyz, x^4, 1)$.

> **Solution:** We can parametrize the surface S by $\vec{r}(s,t) = (s\cos t, s\sin t, 0)$ where $t \in [0, 2\pi]$ and $s \in [0,1]$. Then,
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> $$\vec{r}_s \times \vec{r}_t = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos t & \sin t & 0 \\ -s\sin t & s\cos t & 0 \end{vmatrix} = (0, 0, s\cos^2 t + s\sin^2 t) = (0,0,s).$$
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> Then,
>
> $$\iint_S F(x,y,z) \cdot d\vec{S} = \int_0^1 \int_0^{2\pi} s dt ds = \pi.$$

3. Let S be the collections of points in the xy-plane that satisfy $0 < x^2 + y^2 \leq 1$. Evaluate

$$\iint_S f(x,y,z) dS,$$

where $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2}}$.

> **Solution:** We can parametrize the surface S by $\vec{r}(s,t) = (s\cos t, s\sin t, 0)$ where $t \in [0, 2\pi]$ and $s \in (0,1]$. Then,

$$\vec{r}_s \times \vec{r}_t = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos t & \sin t & 0 \\ -s\sin t & s\cos t & 0 \end{vmatrix} = (0, 0, s\cos^2 t + s\sin^2 t) = (0, 0, s).$$

Since $f(\vec{r}(s,t)) = 1/s$, then

$$\iint_S f(x,y,z)dS = \int_0^1 \int_0^{2\pi} \frac{s}{s} dtds = 2\pi.$$